Appendix A

Review of Kirchhoff Plate Theory

Kirchhoff plate theory and FE

Rigid pavement can be idealized using Kirchhoff theory, which is applicable to thin plates (Cook et al, 1989; Reddy, 1993). In other words, since rigid pavement thickness is very lesser than other two dimensions, transverse shear deformation is insignificant and can be neglected. With this important statement, all stress-strain relations that involved transverse shear deformation are vanished and what remains is the plane stress-strain relation that is shown below in form of matrices (for an isotropic material).

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} E' & E'' & 0 \\ E'' & E' & 0 \\ 0 & 0 & G \end{bmatrix} \cdot \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \alpha \cdot T \\ \alpha \cdot T \\ 0 \end{Bmatrix} \right) \tag{A-12}$$

When α = coefficient of thermal expansion of concrete
μ = Poisson's ratio of concrete
T = temperature differential between top and bottom of concrete

$$E' = \frac{E''}{\mu} = \frac{E}{1-\mu^2} \tag{A-13}$$

$$G = \frac{E}{2 \cdot (1+\mu)} \tag{A-14}$$

Based on stress-strain relations as written in matrix form above, stiffness matrix of concrete slab $[K_p]$ may be derived using the following formula.

$$[K_p] = \int_A [B]^T \cdot [D_k] \cdot [B] \, dA \tag{A-15}$$

When [B] = strain-displacement matrix (will be discussed later)
A = area boundary of an element

$$[D_k] = \begin{bmatrix} D & \mu \cdot D & 0 \\ \mu \cdot D & D & 0 \\ 0 & 0 & \frac{(1-\mu) \cdot D}{2} \end{bmatrix} \tag{A-16}$$

D = flexural rigidity

$$D = \frac{E \cdot t^3}{12 \cdot (1-\mu^2)} \tag{A-17}$$

When t = slab thickness

Winkler foundation and FE

Theoretically, rigid pavement, which is actually a slab on grade, can be approximately considered as one elastic structure supported by a foundation model called Winkler foundation. There are a great many other foundation models available for rigid pavement foundation idealization; however, Winkler foundation is traditionally used and considered as the most effective model. Details of characteristics, advantages, and disadvantages of Winkler foundation will not be discussed at this time. Another name of Winkler foundation is "Dense Liquid" foundation because this foundation simulates the behavior of subgrade or original soil under concrete slab by providing a vertical resistant pressure equal to βw when w is vertical deflection and β is the Winkler foundation modulus (modulus of subgrade reaction). Stiffness matrix of foundation is written below in matrix form.

$$[K_f] = \int_A \beta \cdot [N]^T \cdot [N] \, dA \qquad \text{(A-18)}$$

When [N] = interpolation functions matrix (will be discussed later)
A = area boundary of an element

Discretization into FE and interpolation functions

Since rigid pavement has rectangular geometry, the pavement can be discretized using rectangular linear FE with three degrees of freedom at each node: one vertical displacement, and two horizontal rotations as shown in FigureA-8. In other words, one FE contains twelve degrees of freedom and this means each element has 12x12 stiffness matrix and 12x1 force vector and 12x1 displacement vector.

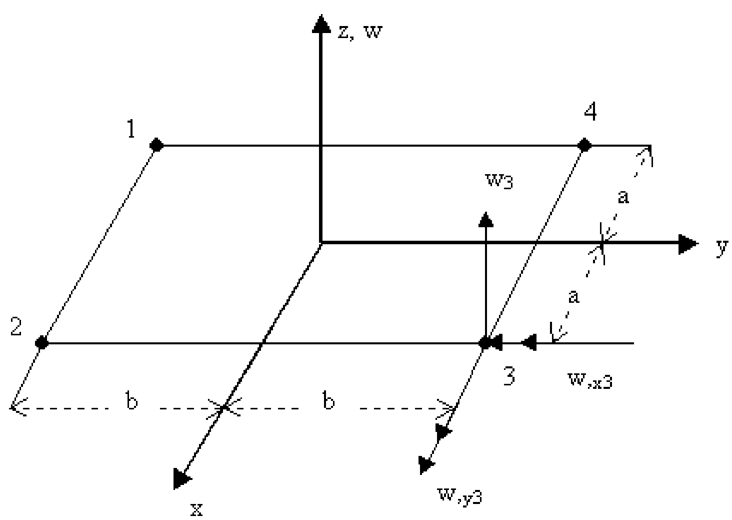

Figure A-8: Twelve-d.o.f. rectangular Kirchhoff plate element with typical d.o.f. shown at node 3

Since Kirchhoff plate elements provide interelement continuity of vertical displacements and rotations in x and y directions, the elements can be considered C^1 elements; therefore, interpolation functions for C^0 elements like Lagrange's interpolation formula may not be applied. Hermitian interpolation function, one of interpolation functions C^1 elements, can be used for this situation (thin plate elements). For an element that has four nodes: 1, 2, 3, and 4, Hermitian interpolation functions can be derived using following formulae.

$$\begin{aligned} w &= N_1 \cdot w_1 + N_{x1} \cdot \theta_{x1} + N_{y1} \cdot \theta_{y1} + N_2 \cdot w_2 + N_{x2} \cdot \theta_{x2} + N_{y2} \cdot \theta_{y2} \\ &+ N_3 \cdot w_3 + N_{x3} \cdot \theta_{x3} + N_{y3} \cdot \theta_{y3} + N_4 \cdot w_4 + N_{x4} \cdot \theta_{x4} + N_{y4} \cdot \theta_{y4} \end{aligned} \tag{A-19}$$

When

$$\begin{bmatrix} N_1 & N_{x1} & N_{y1} \end{bmatrix} = \frac{1}{16} \cdot X_1 Y_1 [X_1 Y_1 - X_2 Y_2 + 2X_1 Y_2 + 2Y_1 Y_2 \quad 2bY_1 Y_2 \quad -2aX_1 X_2] \tag{A-20-1}$$

$$\begin{bmatrix} N_2 & N_{x2} & N_{y2} \end{bmatrix} = \frac{1}{16} \cdot X_2 Y_1 [X_2 Y_1 - X_1 Y_2 + 2X_1 Y_2 + 2Y_1 Y_2 \quad 2bY_1 Y_2 \quad 2aX_1 X_2] \tag{A-20-2}$$

$$\begin{bmatrix} N_3 & N_{x3} & N_{y3} \end{bmatrix} = \frac{1}{16} \cdot X_2 Y_2 [X_2 Y_2 - X_1 Y_1 + 2X_1 Y_2 + 2Y_1 Y_2 \quad -2bY_1 Y_2 \quad 2aX_1 X_2] \tag{A-20-3}$$

$$\begin{bmatrix} N_4 & N_{x4} & N_{y4} \end{bmatrix} = \frac{1}{16} \cdot X_1 Y_2 [X_1 Y_2 - X_2 Y_1 + 2X_1 Y_2 + 2Y_1 Y_2 \quad -2bY_1 Y_2 \quad -2aX_1 X_2] \tag{A-20-4}$$

When $$X_1 = 1 - \frac{x}{a} \tag{A-21-1}$$

$$X_2 = 1 + \frac{x}{a} \tag{A-21-2}$$

$$Y_1 = 1 - \frac{y}{b} \tag{A-21-3}$$

$$Y_2 = 1 + \frac{y}{b} \tag{A-21-4}$$

Now the interpolation functions can be written in matrix form 1x12 as shown below.

$$[N] = [N_1 \;\; N_{x1} \;\; N_{y1} \;\; N_2 \;\; N_{x2} \;\; N_{y2} \;\; N_3 \;\; N_{x3} \;\; N_{y3} \;\; N_4 \;\; N_{x4} \;\; N_{y4}] \tag{A-22}$$

Strain-displacement matrix [B] can also be written in matrix form 3x12 as shown below.

$$[B] = -\begin{bmatrix} \frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_{x1}}{\partial x^2} & \frac{\partial^2 N_{y1}}{\partial x^2} & \cdots & \frac{\partial^2 N_{y4}}{\partial x^2} \\ \frac{\partial^2 N_1}{\partial y^2} & \frac{\partial^2 N_{x1}}{\partial y^2} & \frac{\partial^2 N_{y1}}{\partial y^2} & \cdots & \frac{\partial^2 N_{y4}}{\partial y^2} \\ 2 \cdot \frac{\partial^2 N_1}{\partial x \partial y} & 2 \cdot \frac{\partial^2 N_{x1}}{\partial x \partial y} & 2 \cdot \frac{\partial^2 N_{y1}}{\partial x \partial y} & \cdots & 2 \cdot \frac{\partial^2 N_{y4}}{\partial x \partial y} \end{bmatrix} \tag{A-23}$$

FE of one element

From previous part, stiffness matrix of each element [K_e] (12x12) can be derived as shown below.

$$[K_p] \cdot \{u_p\} + [K_f] \cdot \{u_f\} = \{r_e\} \tag{A-24-1}$$

but $\{u_p\} = \{u_f\} = \{u_e\}$ (A-24-2)

$$[K_e] \cdot \{u_e\} = \{r_e\} \quad \text{(A-24-3)}$$

$$[K_e] = [K_p] + [K_f] \quad \text{(A-24-4)}$$

When $\{u_p\}$ = slab displacement vector
$\{u_f\}$ = foundation displacement vector
$\{u_e\}$ = element displacement vector (12x1)

$$\{u_e\} = \begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ w_3 \\ \theta_{x3} \\ \theta_{y3} \\ w_4 \\ \theta_{x4} \\ \theta_{y4} \end{Bmatrix} \quad \text{(A-25)}$$

$\{r_e\}$ = element force vector (12x1)

$$\{r_e\} = \int_A [B]^T \cdot [D_K] \cdot \{\kappa_o\}\, dA \quad \text{(A-26)}$$

When $\{\kappa_o\} = \begin{bmatrix} \dfrac{\alpha \cdot T}{t} & \dfrac{\alpha \cdot T}{t} & 0 \end{bmatrix}^T$ (A-27)

Global system

Global stiffness matrix and force matrix can be computed based on element stiffness matrix and element force matrix. The concept of generating element stiffness matrix and element force vector into global stiffness matrix and global force vector is exactly the same as the concept of using Boolean matrix that is applicable for C^0 elements but the method is slightly different. This is because each node of a Kirchhoff element has 3 degrees of freedom. This means the element stiffness matrix, which is actually 12x12, can be considered as 4x4 and the element force vector, which is actually 12x1, can be considered 4x1 in order to generate them into global system as shown below.

$$[K_e] = \begin{bmatrix} K_{11(3x3)} & K_{12(3x3)} & K_{13(3x3)} & K_{14(3x3)} \\ K_{21(3x3)} & K_{22(3x3)} & K_{23(3x3)} & K_{24(3x3)} \\ K_{31(3x3)} & K_{32(3x3)} & K_{33(3x3)} & K_{34(3x3)} \\ K_{41(3x3)} & K_{42(3x3)} & K_{43(3x3)} & K_{44(3x3)} \end{bmatrix} \tag{A-28}$$

$$\{r_e\} = \begin{Bmatrix} r_{1(3x1)} \\ r_{2(3x1)} \\ r_{3(3x1)} \\ r_{4(3x1)} \end{Bmatrix} \tag{A-29}$$

Once global stiffness matrix and global force vector are derived, displacement vector of global system can be computed.

$$\{U\}_{3N\times1} = [KG]^{-1}_{3N\times3N} \cdot \{F\}_{3N\times1} \tag{A-30}$$

When {U} = global displacement vector
[KG] = global stiffness matrix
{F} = global force vector
N = number of nodes in global system